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Electromagnetic corrections in the anomaly sector

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Abstract: Chiral perturbation theory in the anomaly sector for $N_f = 2$ is extended to include dynamical photons, thereby allowing a complete treatment of isospin breaking. A minimal set of independent chiral lagrangian terms is determined and the divergence structure is worked out. There are contributions from irreducible and also from reducible one-loop graphs, a feature of ChPT at order larger than four. The generating functional is non anomalous at order e^2p^4, but not necessarily at higher order in e^2. Practical applications to $\gamma\pi \to \pi\pi$ and to the $\pi^0 \to 2\gamma$ amplitudes are considered. In the latter case, a complete discussion of the corrections beyond current algebra is presented including quark mass as well as electromagnetic effects.

Keywords: Electromagnetic Processes and Properties, QCD, Chiral Lagrangians.

Contents

1. Introduction

The precise determination of the masses of the two lightest quarks m_u, m_d in the standard model using chiral perturbation theory (ChPT) requires one to consider small isospin breaking phenomena (see e.g. [1] for a review on the present knowledge of the light quark masses). In this case, it is necessary to account for radiative corrections which induce isospin breaking effects of comparable size to that induced by the quark mass difference $m_d - m_u$. In practice, this amounts to including the photon as a dynamical field, along with those of the quasi-goldstone bosons into the chiral lagrangian. This extension of the conventional ChPT framework [2, 3, 4], in the ordinary sector of ChPT, is due to Urech [5]. Recently, inclusion of dynamical photons was discussed in the sector of the weak non-leptonic interactions [6]. In the present paper we will discuss the odd intrinsic-parity sector of ChPT.

One motivation for this study is the celebrated $\pi^0 \to 2\gamma$ process. In the chiral limit the amplitude is non vanishing and exactly predicted from the ABJ anomaly [7, 8]. Away from this limit, the amplitude receives small corrections from m_u, m_d which are dominated by the isospin breaking difference $m_d - m_u$ as well as electromagnetic corrections. Comparison with high-precision measurements of the π^0 lifetime, such as the PrimEx experiment [9] at JLAB (which plans to achieve a level of precision of less than a percent) could provide a new insight into $m_d - m_u$ and, more generally, furnish a new test of ChPT. An explicit calculation of radiative corrections at one loop was recently performed in ref. [10] for the anomalous amplitude $\gamma\pi^+ \to \pi^+\pi^0$. Our work should complement and provide a check of this calculation. Analogous calculations might also be needed for other precisely determined pionic observables, for instance, the β-decay vector form factor. As far as only pions are concerned, the SU(2) chiral expansion is the most general one.

The plan of the paper is as follows. For kinematical reasons, in the odd intrinsic parity sector the leading chiral order is p^4. At this order, the effective action in this sector is anomalous, i.e. it is not invariant under chiral transformations and its variation under infinitesimal axial transformations must reproduce the Bardeen expression for the anomaly [11]. The explicit form of the anomalous effective action, the Wess-Zumino-Witten action, is known for an arbitrary number of chiral flavours N_f [12]–[15]. Recently, a simpler form was derived for the particular case $N_f = 2$ [16] which is used in the present paper. Using the spurion technique, we have first classified the minimal set of chiral lagrangian terms of order e^2p^4 in the case of SU(2) ChPT. We find that there are eight independent terms and corresponding new chiral coupling-constants k_i^W. We next consider the one-loop contributions to the generating functional of order e^2p^4 in this sector. We observe that while the generating functional is non anomalous at order e^2p^4 this is no longer true at higher order. We compute its divergence structure and the corresponding renormalization of the coupling-constants k_i^W and these results are applied to the $\gamma\pi^+ \to \pi^+\pi^0$ amplitude. Next, we discuss the electromagnetic corrections to the π^0 decay amplitude. In this case, it turns out that the couplings k_i^W make no contributions, the only contributions are generated by one-particle reducible graphs. We derive the expression for the amplitude incorporating the complete set of corrections linear in the quark masses and quadratic in the electric charge. We finally discuss how to exploit this expression by combining resonance saturation estimates and the SU(3) chiral expansion.

2. Chiral lagrangian at order e^2p^4

From the Adler-Bardeen theorem [17] we expect the chiral lagrangian in the odd-intrinsic parity sector at order e^2p^4 to be non anomalous.[1] In order to construct the various terms in the chiral lagrangian we may therefore employ the same type of chiral building blocks as in the ordinary sector.

2.1 Building blocks

Virtual photons induce new chiral lagrangian terms which include factors of the electric charge matrix Q, which for $N_f = 2$ is

$$Q = e \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} . \tag{2.1}$$

[1]The general analysis of ref. [17] of the QED 1PI diagrams shows that all anomalous contributions are generated from diagrams having the one-loop triangle diagram as a sub-graph. Compared to the leading order triangle diagram any contribution of this type is suppressed by a factor e^4.

One can think of these terms as encoding the high-energy part of the photon loops. In order to perform a consistent chiral expansion one must define a chiral counting rule for Q. We will follow the choice proposed in ref. [5]

$$\mathrm{O}(e) \equiv \mathrm{O}(p)\,. \tag{2.2}$$

Next, in order to classify the number of independent chiral lagrangian terms at a given chiral order, one replaces the charge matrix Q by two charge spurions endowed with the transformation properties

$$Q_L \longrightarrow g_L Q_L g_L^\dagger\,, \qquad Q_R \longrightarrow g_R Q_R g_R^\dagger\,. \tag{2.3}$$

Here, we will be interested in classifying the terms of order e^2p^4 in the odd intrinsic-parity sector. A convenient set of building-blocks for the classification of higher order chiral lagrangian terms [18] are those associated with a non-linear representation of the chiral group [19]. We will follow the notation of ref. [18] (see also [20]) which is summarized below. Given the usual unitary matrix U encoding the pseudo-scalar fields and which transforms as $U \to g_R^\dagger U g_L$ one may generate a non-linear representation by setting

$$U = u^2\,, \qquad u \longrightarrow g_R^\dagger u h = h^\dagger u g_L\,. \tag{2.4}$$

A building-block X, then, transforms in the following way under a chiral transformation,

$$X \longrightarrow hXh^\dagger\,. \tag{2.5}$$

A connection Γ_μ may be introduced such that the covariant derivative

$$\nabla_\mu X = \partial_\mu X + [\Gamma_\mu, X]\,, \qquad \text{with } \Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right] \tag{2.6}$$

is also a building block. Another building-block is the field-strength tensor $\Gamma_{\mu\nu}$, generated from the commutator of two covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]X = [\Gamma_{\mu\nu}, X]\,. \tag{2.7}$$

It satisfies the Bianchi Identity, which is important in the odd intrinsic-parity sector [21],

$$\nabla_\alpha \Gamma_{\mu\nu} + \nabla_\mu \Gamma_{\nu\alpha} + \nabla_\nu \Gamma_{\alpha\mu} = 0\,. \tag{2.8}$$

The following standard building-blocks which will appear in our construction

$$u_\mu = u_\mu^\dagger = iu^\dagger D_\mu U u^\dagger = i\left(u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right) \tag{2.9}$$

and

$$f_\pm^{\mu\nu} = uf_L^{\mu\nu}u^\dagger \pm u^\dagger f_R^{\mu\nu} u\,. \tag{2.10}$$

In our construction all the Lorentz indices which appear are contracted with the antisymmetric ϵ tensor. It is therefore not necessary to consider more than one derivative acting on

	u_μ	$f_{\pm\mu\nu}$	$Q_\pm$	$Q_{\pm\mu}$
P	$-u^\mu$	$\pm f_\pm^{\mu\nu}$	$\pm Q_\pm$	$\pm Q_\pm^\mu$
C	u_μ^T	$\mp f_{\pm\mu\nu}^T$	$\mp Q_\pm^T$	$\mp Q_{\pm\mu}^T$

Table 1: Transformation properties of the chiral building-blocks under parity (P) and charge conjugation (C).

one building-block because of relation (2.7). Also $\nabla_\mu u_\nu$ need not be considered separately because of the relation

$$\nabla_\mu u_\nu - \nabla_\nu u_\mu = -f_{-\mu\nu}\,. \tag{2.11}$$

We also recall the following relation

$$\Gamma_{\mu\nu} = \frac{1}{4}[u_\mu, u_\nu] - \frac{i}{2} f_{+\mu\nu} \tag{2.12}$$

which indicates that $\Gamma_{\mu\nu}$ need not be independently considered.

From the charge spurions, one can form the building blocks

$$Q_\pm = uQ_L u^\dagger \pm u^\dagger Q_R u \tag{2.13}$$

and one can also consider their covariant derivatives. We will, instead, use the following building blocks

$$Q_\pm^\mu = u c_L^\mu Q_L u^\dagger \pm u^\dagger c_R^\mu Q_R u \tag{2.14}$$

in which the derivatives acting on the charge spurions are defined as [5]

$$c_L^\mu Q_L = \partial^\mu Q_L - i[l_\mu, Q_L]\,, \qquad c_R^\mu Q_R = \partial^\mu Q_R - i[r_\mu, Q_R]\,. \tag{2.15}$$

They are related to covariant derivatives by the following relation

$$Q_\pm^\mu = \nabla^\mu Q_\pm + \frac{i}{2}[u^\mu, Q_\mp]\,. \tag{2.16}$$

This completes the list of the chiral building-blocks needed in our construction. Their transformation properties under parity and charge conjugation are collected in table 1.

2.2 Classification of a minimal set of independent terms

Our purpose is to construct a complete set of chiral lagrangian terms, which are chirally invariant, and of order e^2p^4 in the odd intrinsic-parity sector. We must then have four lorentz indices contracted with the ϵ tensor and two insertions of the charge matrix Q. One can a priori form seven independent types of terms of this kind

$$\begin{aligned}
&1)\ \epsilon^{\mu\nu\alpha\beta} QQ u_\mu u_\nu u_\alpha u_\beta\,, \qquad 2)\ \epsilon^{\mu\nu\alpha\beta} QQ f_{\mu\nu} u_\alpha u_\beta\,, \qquad 3)\ \epsilon^{\mu\nu\alpha\beta} QQ f_{\mu\nu} f_{\alpha\beta}\,,\\
&4)\ \epsilon^{\mu\nu\alpha\beta} Q_\mu Q u_\nu u_\alpha u_\beta\,, \qquad 5)\ \epsilon^{\mu\nu\alpha\beta} Q_\mu Q u_\nu f_{\alpha\beta}\,, \qquad 6)\ \epsilon^{\mu\nu\alpha\beta} Q_\mu Q_\nu u_\alpha u_\beta\,,\\
&7)\ \epsilon^{\mu\nu\alpha\beta} Q_\mu Q_\nu f_{\alpha\beta}\,.
\end{aligned} \tag{2.17}$$

Here we have not specified the $\pm$ subscripts and all the possible independent orderings and trace structures are to be worked out later. Using eq. (2.16) we see that by using integration by parts we can always express the structures 6) and 7) in terms of the others, so only the first five structures are needed.

We specialize to the case of SU(2) ChPT in this section, which allows one to discuss electromagnetic corrections for processes involving pions. For 2×2 matrices the following identity relates the anti-commutator to traces

$$\{A, B\} = A\langle B\rangle + B\langle A\rangle + \langle AB\rangle - \langle A\rangle\langle B\rangle \,. \tag{2.18}$$

In connection with this identity, one notes that the product of three matrices can be written in terms of anti-commutators,

$$ABC = \frac{1}{2}\left(\{A, B\}C + A\{B, C\} - \{B, AC\}\right). \tag{2.19}$$

For classification purposes, it is thus not necessary to consider traces of products of more than three matrices. In the case of SU(2) the trace of the charge matrix does not vanish but we can restrict ourselves to $\langle Q_L\rangle = \langle Q_R\rangle$ (following ref. [22]) i.e.

$$\langle Q_-\rangle = 0 \,. \tag{2.20}$$

Also we will assume that $\langle \partial_\mu Q_L\rangle = \langle \partial_\mu Q_R\rangle = 0$, i.e.

$$\langle Q_{\pm\mu}\rangle = 0 \,. \tag{2.21}$$

Concerning the external vector and axial-vector fields we must consider the case $\langle f_{+\mu\nu}\rangle \neq 0$ but we can restrict ourselves to $\langle f_{-\mu\nu}\rangle = 0$. Let us now start the classification of independent terms.

- Terms of type (1)

  Making use of the relations (2.18) and (2.19) it is easy to see that there is a single term of this kind,

$$\hat{o}_1^W \equiv i\epsilon^{\mu\nu\alpha\beta}\langle Q_+\rangle\langle Q_- u_\mu\rangle\langle u_\nu u_\alpha u_\beta\rangle \,. \tag{2.22}$$

  (the factor of i ensures hermiticity)

- Terms of type (2)

  We begin by considering terms containing Q_+, Q_-, $f_{+\mu\nu}$. One finds that there are two independent terms of this type, which we choose as

$$\hat{o}_2^W \equiv \epsilon^{\mu\nu\alpha\beta}\langle [Q_+, Q_-]u_\alpha u_\beta\rangle\langle f_{+\mu\nu}\rangle \,, \qquad \hat{o}_3^W \equiv \epsilon^{\mu\nu\alpha\beta}\langle Q_+\rangle\langle [Q_-, f_{+\mu\nu}]u_\alpha u_\beta\rangle \,. \tag{2.23}$$

  Next, we consider the terms containing Q_+, Q_+, $f_{-\mu\nu}$, we find only one independent term of this kind

$$\hat{o}_4^W \equiv \epsilon^{\mu\nu\alpha\beta}\langle Q_+^2 u_\alpha\rangle\langle f_{-\mu\nu} u_\beta\rangle \,. \tag{2.24}$$

  Finally, we consider the terms containing Q_-, Q_-, $f_{-\mu\nu}$: all possible terms of this kind which are C-even turn out to vanish.

- Terms of type (3)

  We consider first the combinations $Q_+ Q_- f_{+\mu\nu} f_{+\alpha\beta}$. There is only one C-even double trace term

$$\hat{o}_5^W \equiv i\epsilon^{\mu\nu\alpha\beta}\langle [Q_+, f_{+\mu\nu}]Q_-\rangle\langle f_{+\alpha\beta}\rangle \tag{2.25}$$

and no C-even triple trace terms. Next, the combination $Q_+Q_-f_{-\mu\nu}f_{-\alpha\beta}$ gives rise to no contribution because we restrict ourselves to $\langle f_{-\mu\nu}\rangle = 0$. What remains to be considered are the two combinations $Q_+Q_+f_{+\mu\nu}f_{-\alpha\beta}$ and $Q_-Q_-f_{+\mu\nu}f_{-\alpha\beta}$. Only one invariant is generated from these

$$\hat{o}_6^W \equiv i\epsilon^{\mu\nu\alpha\beta}\langle Q_+\rangle\langle [Q_+, f_{+\mu\nu}]f_{-\alpha\beta}\rangle \,. \tag{2.26}$$

- Terms of type (4)

  Using parity invariance one needs consider the combination $Q_{+\mu}Q_+$ or $Q_{-\mu}Q_-$, it turns out that no C-even term of type (4) can be formed.

- Terms of type (5)

  One finds four independent terms of this type

$$\hat{o}_7^W \equiv i\epsilon^{\mu\nu\alpha\beta}\langle [Q_{+\mu}, Q_+]u_\nu\rangle\langle f_{+\alpha\beta}\rangle \,, \qquad \hat{o}_8^W \equiv i\epsilon^{\mu\nu\alpha\beta}\langle [Q_{-\mu}, Q_-]u_\nu\rangle\langle f_{+\alpha\beta}\rangle \tag{2.27}$$

  and

$$\hat{o}_9^W \equiv \epsilon^{\mu\nu\alpha\beta}\langle [Q_{+\mu}, u_\nu]f_{+\alpha\beta}\rangle\langle Q_+\rangle \,, \qquad \hat{o}_{10}^W \equiv \epsilon^{\mu\nu\alpha\beta}\langle [Q_{-\mu}, u_\nu]f_{-\alpha\beta}\rangle\langle Q_+\rangle \,. \tag{2.28}$$

  The last two terms need not, however, be considered because using integration by parts they can be related to terms already considered and to terms containing $\langle Q_{+\mu}\rangle$ which was assumed to vanish. Finally, one must worry about contact terms, i.e. invariants involving external fields only. No such terms can be formed here which respect P and C invariance.

In conclusion, we find that there are eight independent chiral lagrangian terms of order e^2p^4, in the odd intrinsic-parity sector. We may label the associated chiral coupling constants as k_i^W by analogy with the couplings k_i introduced for the SU(2) chiral lagrangian at order e^2p^2 [22]. The resulting chiral lagrangian is summarized below,

$$\begin{aligned}
\mathcal{L}^W_{e^2p^4} = \epsilon^{\mu\nu\alpha\beta}\big\{ & k_1^W\, i\langle Q_+\rangle\langle Q_-u_\mu\rangle\langle u_\nu u_\alpha u_\beta\rangle + k_2^W\,\langle [Q_+, Q_-]u_\alpha u_\beta\rangle\langle f_{+\mu\nu}\rangle + \\
& + k_3^W\,\langle Q_+\rangle\langle [Q_-, f_{+\mu\nu}]u_\alpha u_\beta\rangle + k_4^W\,\langle Q_+^2 u_\alpha\rangle\langle f_{-\mu\nu}u_\beta\rangle + \\
& + k_5^W\, i\langle [Q_+, f_{+\mu\nu}]Q_-\rangle\langle f_{+\alpha\beta}\rangle + k_6^W\, i\langle Q_+\rangle\langle [Q_+, f_{+\mu\nu}]f_{-\alpha\beta}\rangle + \\
& + k_7^W\, i\langle [Q_{+\mu}, Q_+]u_\nu\rangle\langle f_{+\alpha\beta}\rangle + k_8^W\, i\langle [Q_{-\mu}, Q_-]u_\nu\rangle\langle f_{+\alpha\beta}\rangle\big\} \,.
\end{aligned} \tag{2.29}$$

3. Divergence structure

In order to determine the divergent parts of the generating functional $S^W_{e^2p^4}$ we can proceed by analogy with the general discussion of Bijnens, Colangelo and Ecker (BCE) [23] of the $O(p^6)$ divergence structure of ChPT. Since we concern ourselves with the odd intrinsic-parity sector here, only one-loop diagrams will contribute which have one vertex from the WZW action $S^W_{p^4}$. The one-particle irreducible diagrams are shown in figure 1. Di-

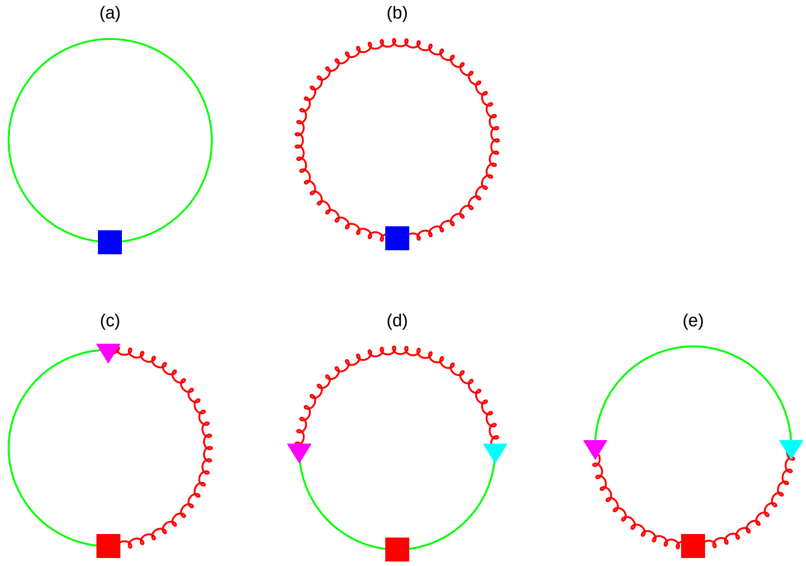

Figure 1: One particle irreducible graphs which contribute to the $\mathrm{O}(p^6)$ and the $\mathrm{O}(e^2p^4)$ divergences in the anomalous sector: a triangle denotes an $\mathrm{O}(p^2)$ vertex a box denotes an $\mathrm{O}(p^4)$ vertex. The solid line represents the (full) pion propagator and the curly line the full photon propagator.

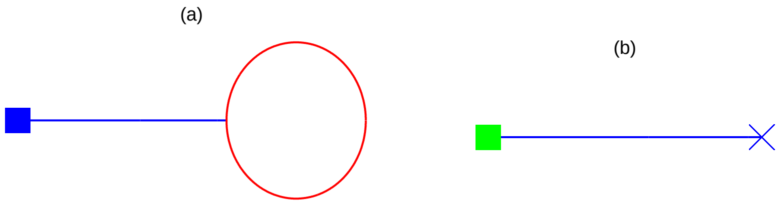

Figure 2: Sample one particle reducible graphs which can contribute to the $\mathrm{O}(p^6)$ and the $\mathrm{O}(e^2p^4)$ divergences. A square denotes a vertex from the WZW lagrangian, a cross a vertex from the $\mathrm{O}(p^4)$ lagrangian.

agram (a) does not contain the photon explicitly but a contribution proportional to e^2 can get generated from the $\pi^+ - \pi^0$ mass difference, also proportional to the low-energy coupling C. The divergent contribution from this diagram when $e = 0$ was evaluated previously for generic N_f and traceless external vector or axial-vector fields [21, 24, 25, 26] and in the particular case of $N_f = 2$ and non-traceless external vector fields in ref. [27].

As pointed out by BCE it is also necessary to consider one-particle reducible graphs. Such graphs are shown in figure 2. The origin of these extra divergences is that the $\mathrm{O}(p^4)$ chiral lagrangian cancels the $\mathrm{O}(p^4)$ one-loop divergences only up to terms which vanish upon using the equations of motion. The form of these depends on the choice of independent terms in the $\mathrm{O}(p^4)$ chiral lagrangian. We will next display the expressions for the various vertices involved and then give the results of computing the diagrams of figures 1 and 2.

3.1 $O(p^2)$ vertices

One starts from the $O(p^2)$ chiral lagrangian including dynamical photon fields [5],

$$\begin{aligned}\mathcal{L}_{p2} = & \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi^\dagger U + U^\dagger \chi\rangle + C\langle Q_R U Q_L U^\dagger\rangle - \\ & -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} - \frac{1}{r2\lambda}(\partial^\mu A_\mu)^2 \,.\end{aligned} \tag{3.1}$$

The photon field A_μ also appears in the covariant derivative

$$D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu + iA_\mu(-Q_R U + UQ_L)\,. \tag{3.2}$$

The fluctuations of the pion field around the classical configuration may be defined by setting [4]

$$U = u_{\rm cl}\,\exp(i\xi)\,u_{\rm cl}\,, \qquad \xi = \sum_1^{N_f^2-1}\frac{\xi^a\lambda^a}{F}\,. \tag{3.3}$$

The terms linear in ξ can then be made to vanish by imposing that the equation of motion be satisfied by the classical configuration,

$$\left(\nabla_\mu u^\mu - \frac{i}{2}\hat{\chi}_- - \frac{iC}{F^2}[Q_+,Q_-]\right)_{u=u_{\rm cl}} = 0\,, \qquad \hat{\chi}_- = \chi_- - \frac{1}{N_f}\langle\chi_-\rangle\,. \tag{3.4}$$

Next, from the terms quadratic in ξ one defines the pion propagator G_Δ^{ab} (we use the same notation as BCE). It satisfies the following equation (in euclidean space)

$$(-d_\mu d_\mu + \sigma)G_\Delta(x,y) = \delta(x-y)\,, \tag{3.5}$$

where

$$d_\mu = \partial_\mu + \gamma_\mu\,, \qquad (\gamma_\mu)^{ab} = -\frac{1}{2}\langle\Gamma_\mu[\lambda^a,\lambda^b]\rangle \tag{3.6}$$

and

$$\begin{aligned}\sigma^{ab} = & \frac{1}{8}\langle[u_\mu,\lambda^a][u_\mu,\lambda^b]\rangle + \frac{1}{8}\langle\{\lambda^a,\lambda^b\}\chi_+\rangle - \\ & -\frac{C}{4F^2}\langle[Q_+,\lambda^a][Q_+,\lambda^b] - [Q_-,\lambda^a][Q_-,\lambda^b]\rangle\,.\end{aligned} \tag{3.7}$$

Concerning the photon field, its classical part is included in the external field v_μ (which is not constrained by an equation of motion) and the quantum part is designated by A_μ. If we restrict ourselves to terms quadratic in the electric charge we will only need the free propagator which, for an arbitrary gauge parameter λ reads

$$G^{\mu\nu}(x) = \delta^{\mu\nu}G_0(x) + (1-\lambda)\int dz\,\partial^\mu\partial^\nu G_0(z)G_0(x-z)\,. \tag{3.8}$$

G_0 being the massless scalar free-field propagator. The remaining terms quadratic in the fluctuating fields yield the two $O(p^2)$ vertices (all vertices below are displayed in euclidean space)

$$\begin{aligned}
\mathcal{L}^{(2)}_{\xi A} &= -\frac{F}{2}\left((A_\mu\, d_\mu \xi^a)\, \langle \lambda^a Q_- \rangle + (A_\mu \xi^a) \frac{i}{2}\, \langle \lambda^a [u_\mu, Q_+] \rangle \right), \\
\mathcal{L}^{(2)}_{A} &= \frac{F^2}{2} A_\mu\, \langle u_\mu Q_- \rangle\,, \\
\mathcal{L}^{(2)}_{AA} &= -\frac{F^2}{4} A_\mu A_\mu\, \langle Q_-^2 \rangle\,.
\end{aligned} \tag{3.9}$$

3.2 O(p^4) vertices

Let us now turn to the vertices generated from the O(p^4) Wess-Zumino-Witten action. In the case of two flavours, to which we will restrict ourselves, the action is local and Kaiser [16] has recently derived a remarkably simple expression for the lagrangian,

$$\mathcal{L}^W_{p^4} = \kappa\, \epsilon^{\mu\nu\alpha\beta} \left(\langle U^\dagger r_\mu U l_\nu - r_\mu l_\nu + i\Sigma_\mu (U^\dagger r_\nu U + l_\nu) \rangle \langle v_{\alpha\beta} \rangle + \frac{2}{3} \langle \Sigma_\mu \Sigma_\nu \Sigma_\alpha \rangle \langle v_\beta \rangle \right) \tag{3.10}$$

with

$$\kappa = -\frac{N_c}{32\pi^2} \tag{3.11}$$

and

$$\Sigma_\mu = U^\dagger \partial_\mu U\,, \qquad l_\mu = v_\mu - a_\mu\,, \qquad r_\mu = v_\mu + a_\mu\ , \qquad \epsilon^{0123} = -1\,. \tag{3.12}$$

Performing the fluctuations we first obtain the vertices which are linear in the quantum fields (calling $\hat{\epsilon}$ the ϵ tensor rotated to euclidean space),

$$\begin{aligned}
\mathcal{L}^W_{\xi} &= \frac{1}{4F} \kappa\, \hat{\epsilon}_{\mu\nu\alpha\beta} (\xi^a)\, \langle \lambda^a (2i u_\mu u_\nu + f_{+\mu\nu}) \rangle\, \langle f_{+\alpha\beta} \rangle\,, \\
\mathcal{L}^W_{A} &= \frac{2}{3} \kappa\, \hat{\epsilon}_{\mu\nu\alpha\beta} (A_\mu)\, \langle \Sigma_\nu \Sigma_\alpha \Sigma_\beta \rangle\, \langle Q \rangle\,.
\end{aligned} \tag{3.13}$$

Then, we obtain the three vertices which are quadratic,

$$\begin{aligned}
\mathcal{L}^W_{\xi\xi} &= \frac{i}{4F^2} \kappa\, \hat{\epsilon}_{\mu\nu\alpha\beta} (d_\mu \xi^a \xi^b) \left\langle [\lambda^a, \lambda^b] u_\nu \right\rangle \langle f_{+\alpha\beta} \rangle\,, \\
\mathcal{L}^W_{\xi A} &= \frac{1}{2F} \kappa\, \hat{\epsilon}_{\mu\nu\alpha\beta} \{ -(A_\mu \xi^a)\, \langle \lambda^a (Q_{+\nu} + i[Q_-, u_\nu]) \rangle\, \langle f_{+\alpha\beta} \rangle + \\
&\qquad + (\partial_\mu A_\nu \xi^a)\, [\langle \lambda^a Q_+ \rangle\, \langle f_{+\alpha\beta} \rangle + \langle \lambda^a (f_{+\alpha\beta} + 2i u_\alpha u_\beta) \rangle\, \langle Q_+ \rangle] \}\,, \\
\mathcal{L}^W_{AA} &= 2\kappa\, \hat{\epsilon}_{\mu\nu\alpha\beta} (A_\nu \partial_\alpha A_\beta)\, (\langle u_\mu Q_+ \rangle\, \langle Q_+ \rangle - 2\, \langle a_\mu (Q_L + Q_R) \rangle\, \langle Q_+ \rangle)\,.
\end{aligned} \tag{3.14}$$

One observes that the vertices with either one or two $\xi' s$ involve only canonical chiral building blocks. This implies that the contribution from graph (a) of figure 1 will obey ordinary chiral Ward identities [24, 25, 26]. This property remains true for graphs (c) and (d). In contrast, the vertices which contain only A_μ generate non-invariant pieces. It turns out, as we will see below, that these non-invariant pieces do not affect the generating functional at order $e^2 p^4$ (except, however, for the Coulomb term). The Adler-Bardeen non-renormalization theorem [17] is therefore satisfied in the effective theory at this order. We have not investigated how the Adler-Bardeen discussion is compatible with the effective theory at higher order in e^2.

3.3 One-loop 1PI diagrams

Let us now consider each diagram if figure 1 in turn.

• diagram (a):

This diagram was considered in refs. [21, 24, 25, 26] and the divergent part was extracted for generic N_f but traceless external fields and in ref. [27] for the particular case of two flavours and non-traceless external vector fields. We consider the latter situation here and want to extend the calculation to include the terms proportional to e^2C. The contribution from this diagram to the generating functional in euclidean space is straightforward to obtain,

$$-Z^{(a)} = \frac{i}{4F^2}\kappa\,\hat{\epsilon}_{\mu\nu\alpha\beta}\int dx\, d^x_\mu G^{ab}_\Delta(x, y=x)\left\langle [\lambda^a, \lambda^b] u_\nu \right\rangle \langle f_{+\alpha\beta}\rangle\,, \tag{3.15}$$

The divergent part of the propagator can be obtained by heat-kernel methods (see ref. [23] and references therein) in dimensional regularization

$$[d^x_\mu G_\Delta(x, y=x)]_{\rm div} = (-2)\frac{1}{16\pi^2(d-4)} d^x_\mu a_1(x, y=x)\,. \tag{3.16}$$

In the coincidence limit, the derivative of the heat-kernel coefficient a_1 is easily worked out to be

$$d_\mu a_1 = -\frac{1}{6}(\partial_\lambda\gamma_{\lambda\mu} + [\gamma_\lambda, \gamma_{\lambda\mu}]) - \frac{1}{2}(\partial_\mu\sigma + [\gamma_\mu, \sigma])\,. \tag{3.17}$$

The terms containing σ will not contribute to the divergence because they are symmetric in the a, b indices. Performing the sum over a, b and returning to Minkowski space one finds that the divergence can be written in terms of a lagrangian,

$$\mathcal{L}^{(a)}_{\rm div} = \frac{1}{16\pi^2(d-4)}\frac{-2\kappa}{3F^2} i\epsilon^{\mu\nu\alpha\beta}\langle\nabla^\lambda\Gamma_{\lambda\mu}u_\nu\rangle\langle f_{+\alpha\beta}\rangle \tag{3.18}$$

in agreement with the previous results [21, 24, 25, 26]. At this point, one notes that no term proportional to e^2C appears. Such terms appear only at a later stage, upon using the equation of motion. Therefore, they depend on the choice made for the independent chiral lagrangian terms in $\mathcal{L}^W_{p^6}$. We will use the basis of ref. [27]. For convenience, we reproduce below the list of terms from this basis which are relevant for the divergence:

$$\begin{aligned}
o^W_6 &\equiv \epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\chi_- u_\alpha u_\beta\rangle\,, & o^W_7 &\equiv i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\chi_-\rangle\,,\\
o^W_8 &\equiv i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\rangle\langle\chi_-\rangle\,, & o^W_9 &\equiv i\epsilon^{\mu\nu\alpha\beta}\left\langle f^\gamma_{+\mu}\right\rangle\langle h_{\gamma\nu}u_\alpha u_\beta\rangle\,,\\
o^W_{10} &\equiv i\epsilon^{\mu\nu\alpha\beta}\left\langle f^\gamma_{+\mu}\right\rangle\langle f_{-\gamma\nu}u_\alpha u_\beta\rangle\,, & o^W_{11} &\equiv \epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\left\langle f^\gamma_{+\alpha}h_{\gamma\beta}\right\rangle\,,\\
o^W_{12} &\equiv \epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\left\langle f^\gamma_{+\alpha}f_{-\gamma\beta}\right\rangle\,, & o^W_{13} &\equiv \epsilon^{\mu\nu\alpha\beta}\left\langle\nabla_\gamma f^\gamma_{+\mu}\right\rangle\langle f_{+\nu\alpha}u_\beta\rangle\,.
\end{aligned} \tag{3.19}$$

Let us also introduce two additional chiral lagrangian terms which appear if one does not make use of the equations of motion

$$\begin{aligned}
o^W_{E1} &\equiv i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\nabla^\gamma u_\gamma u_\alpha u_\beta\rangle\,,\\
o^W_{E2} &\equiv \epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\nabla^\gamma u_\gamma f_{+\alpha\beta}\rangle\,.
\end{aligned} \tag{3.20}$$

Not using the equation of motion, one finds that the divergence from graph (a) has the following expansion

$$\mathcal{L}^{(a)}_{\text{div}} = \frac{-1}{16\pi^2(d-4)}\frac{\kappa}{6F^2}\left(o^W_{E1} - o^W_{E2} - o^W_9 + 3o^W_{10} - 2o^W_{11} + 2o^W_{13}\right). \tag{3.21}$$

Using the equation of motion eq. (3.4), one obtains, firstly, the part which does not depend on e^2,

$$\mathcal{L}^{(a)}_{\text{div1}} = \frac{-1}{16\pi^2(d-4)}\frac{\kappa}{6F^2}\left(-\frac{1}{2}o^W_6 - \frac{1}{2}o^W_7 + \frac{1}{4}o^W_8 - o^W_9 + 3o^W_{10} - 2o^W_{11} + 2o^W_{13}\right). \tag{3.22}$$

This reproduces the result of ref. [27] but one must keep in mind that this does not represent the complete divergence because of the presence of 1PR contributions to be discussed below. Secondly, a part proportional to e^2C appears, which can be expressed over the appropriate invariants $\hat{o}^W_i$ introduced above,

$$\mathcal{L}^{(a)}_{\text{div2}} = \frac{-1}{16\pi^2(d-4)}\frac{\kappa}{6}\frac{C}{F^4}(-\hat{o}^W_2 + \hat{o}^W_5). \tag{3.23}$$

- diagram (b):

This diagram is easily seen to vanish since the vertex function is antisymmetric in two Lorentz indices which must be contracted with those of the photon propagator which is symmetric.

- diagram (c):

In order to obtain the divergent part of this diagram one uses the short distance expansion of the pion propagator $G_\Delta(x,y)$ obtained from its heat kernel representation,

$$G_\Delta(x,y) = G_0(x-y)a_0(x,y) + \cdots. \tag{3.24}$$

Only the first term will contribute to the divergence in the present case. One also needs the divergent part in the product of two free propagators

$$[G_0(x-y)^2]_{\text{div}} = \frac{-2}{16\pi^2(d-4)}\delta(x-y) \tag{3.25}$$

and the derivative of the preceding expression. Using the vertices from eqs. (3.9) and (3.14) a small calculation gives the divergence in the following form in the Feynman gauge, $\lambda = 1$

$$\begin{aligned}\mathcal{L}^{(c)}_{\text{div}} = \frac{1}{16\pi^2(d-4)}\frac{1}{2}\kappa\,\epsilon^{\mu\nu\alpha\beta}\Big\{&\langle f_{+\alpha\beta}\rangle\Big\langle(\nabla_\mu Q_- - i[u_\mu, Q_+])\Big(\nabla_\nu Q_+ - \frac{i}{2}[u_\nu, Q_-]\Big)\Big\rangle + \\ &+ \langle(f_{+\alpha\beta} + 2iu_\alpha u_\beta)\{Q_+, Y_{\mu\nu}\}\rangle\Big\},\end{aligned} \tag{3.26}$$

where

$$Y_{\mu\nu} = -\frac{1}{2}[\Gamma_{\mu\nu}, Q_-] - \frac{i}{4}[f_{-\mu\nu}, Q_+] - \frac{i}{2}[u_\mu, \nabla_\nu Q_+]. \tag{3.27}$$

This can be expressed in terms of our set of invariants

$$\mathcal{L}^{(c)}_{\text{div}} = \frac{-1}{16\pi^2(d-4)}\frac{\kappa}{8}\left(4\hat{o}^W_2 - \hat{o}^W_3 - 2\hat{o}^W_4 + 2\hat{o}^W_7 - 2\hat{o}^W_8\right). \tag{3.28}$$

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• diagram (d):

In this case, the divergence arises from the product of three free propagators having two or more derivatives acting on them. The formula which is needed here is

$$[G_0(z-x)G_0(z-y)\partial_\mu\partial_\nu G_0(x-y)]_{\text{div}} = \frac{1}{2}\frac{1}{16\pi^2(d-4)}\delta_{\mu\nu}\delta(x-y)\delta(x-z)\,. \tag{3.29}$$

It is then not difficult to find the divergence from this diagram, in the Feynman gauge

$$\mathcal{L}^{(d)}_{\text{div}} = \frac{1}{16\pi^2(d-4)}\frac{i}{4}\kappa\,\epsilon^{\mu\nu\alpha\beta}\left\langle f_{+\alpha\beta}\right\rangle\left\langle\left(\nabla_\mu Q_- - \frac{i}{2}[u_\mu,Q_+]\right)[u_\nu,Q_-]\right\rangle, \tag{3.30}$$

which, in terms of the basis of invariants read

$$\mathcal{L}^{(d)}_{\text{div}} = \frac{-1}{16\pi^2(d-4)}\frac{\kappa}{8}\left(-2\hat{o}^W_2 + 2\hat{o}^W_8\right). \tag{3.31}$$

The last diagram (e) is found to be convergent. In any case, it is quartic in the electric charge, which is beyond the accuracy to which we restrict ourselves here.

To summarize, the divergence from the 1PI diagrams is contained in eqs. (3.23), (3.26) and (3.30). An alternative method for determining this part of the divergence is to express the one-loop generating functional as the trace-log of an operator. We have performed the computation in this way also and recovered the results of the diagrammatic approach. The diagrammatic method makes it somewhat easier to make use of gauges other than the Feynman gauge. It is not difficult, from the expression of the photon propagator in an arbitrary gauge, eq. (3.8) to determine the additional divergent part for $\lambda \neq 1$

$$\mathcal{L}^{(c+d)}_{\text{div},\lambda} = (1-\lambda)\frac{-1}{16\pi^2(d-4)}\frac{\kappa}{16}\left(-4\hat{o}^W_2 - \hat{o}^W_5 + 2\hat{o}^W_7 + 6\hat{o}^W_8\right). \tag{3.32}$$

In the following, we will restrict ourselves to $\lambda = 1$.

3.4 One-loop 1PR diagrams

Let us now consider the contributions to the $\mathrm{O}(p^6)$ and $\mathrm{O}(e^2p^4)$ divergence generated by one-particle reducible one-loop diagrams (see figure 2). As is explained by BCE [23] such contributions arise because the divergences of the generating functional at one loop are canceled by the contributions from the chiral $\mathrm{O}(p^4)$ action up to terms proportional to the equation of motion. If one uses the SU(2) chiral lagrangian of ref. [3] extended to include dynamical photons by Knecht and Urech (KU) [22], one finds the following results for these EOM terms,

$$S^{\text{EOM}} \equiv S^{\text{div}}_{p^4} + S^{KU}_{p^4} = \frac{-1}{16\pi^2(d-4)}\int d^4x\left(\frac{i}{2}\left\langle\chi^- X_2\right\rangle - \frac{iF^2}{8}\left\langle[Q_+,Q_-]X_2\right\rangle\right) \tag{3.33}$$

with

$$X_2 = \nabla^\mu u_\mu - \frac{i}{2}\hat{\chi}_- - \frac{iC}{F^2}[Q_+,Q_-]\,. \tag{3.34}$$

The one ξ vertex resulting from the lagrangian of eq. (3.33) reads

$$\mathcal{L}^{\text{EOM}}_\xi = \frac{-1}{16\pi^2(d-4)}(-d^2+\sigma)^{ab}\xi^b\left(\frac{i}{2F}\left\langle\lambda^a\hat{\chi}_-\right\rangle - \frac{iF}{8}\left\langle\lambda^a[Q_+,Q_-]\right\rangle\right). \tag{3.35}$$

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Together with the one ξ vertex from $\mathcal{L}^W_{p^4}$, eq. (3.35), this produces the following local divergent terms of order p^6 and e^2p^4 from the diagrams of figure 2

$$\mathcal{L}^{(f+g)}_{\text{div}} = \frac{-1}{16\pi^2(d-4)}\left[\frac{\kappa}{2F^2}\left(-o^W_6+\frac{1}{2}o^W_7-\frac{1}{4}o^W_8\right)+\frac{\kappa}{8}\left(-\hat{o}^W_2-\frac{1}{2}\hat{o}^W_5\right)\right]. \tag{3.36}$$

Finally, the UV divergences of the Green's functions at order e^2p^4 can be absorbed into the bare coupling constants k^W_i. We will adopt the same convention as ref. [27] for the relation between bare and renormalized couplings (which differs from the normalization adopted in the ordinary p^6 sector [23]),

$$k^W_{i,\text{bare}} = (c\mu)^{d-4}\left(\frac{1}{16\pi^2(d-4)}\beta_i + k^W_{i,r}(\mu)\right), \tag{3.37}$$

with $\log c = -(\log 4\pi + \Gamma'(1)+1)$. Collecting the various pieces in the divergences we find that the coefficients β_i have the following values in the Feynman gauge

$$\beta_1 = \beta_6 = \beta_8 = 0 \tag{3.38}$$

and

$$\begin{aligned}\beta_2 &= \left(\frac{1}{8}-\frac{1}{6}\frac{C}{F^4}\right)\kappa\,, \qquad \beta_3 = -\frac{1}{8}\kappa\,, \qquad \beta_4 = -\frac{1}{4}\kappa\,,\\ \beta_5 &= \left(-\frac{1}{16}+\frac{1}{6}\frac{C}{F^4}\right)\kappa\,, \qquad \beta_7 = \frac{1}{4}\kappa\,.\end{aligned} \tag{3.39}$$

We also quote the values of the analogous parameters η_i associated with the $\mathrm{O}(p^6)$ couplings c^W_i taking into account the 1PI as well as the 1PR contributions,

$$\begin{aligned}\eta_6 &= -\frac{7}{12}\frac{\kappa}{F^2}\,, \qquad \eta_7 = \frac{1}{6}\frac{\kappa}{F^2}\,, \qquad \eta_8 = -\frac{1}{12}\frac{\kappa}{F^2}\,,\\ \eta_9 &= -\frac{1}{6}\frac{\kappa}{F^2}\,, \qquad \eta_{10} = \frac{1}{2}\frac{\kappa}{F^2}\,, \qquad \eta_{11} = -\frac{1}{3}\frac{\kappa}{F^2}\,,\\ \eta_{12} &= 0\,, \qquad \eta_{13} = \frac{1}{3}\frac{\kappa}{F^2}\,,\end{aligned} \tag{3.40}$$

and $\eta_i = 0,\ i = 1,\ldots,5$.

3.5 Application to $\gamma\pi \to \pi\pi$

These results can be applied to the computation of radiative corrections at one loop to various processes in the anomaly sector. Taking into account the tree level contributions from $\mathcal{L}^W_{e^2p^4}$ enables one to express the result in a finite and scale independent way in terms of a minimal number of coupling constants. For instance, let us consider the process $\gamma\pi^+ \to \pi^+\pi^0$ for which the one-loop radiative corrections have already been calculated [10]. We follow the notation of this reference and denote the amplitude for this process by $F^{3\pi}$,

$$\langle 0|J^{em}_\mu(0)|\pi^+\pi^-\pi^0\rangle = i\epsilon^{\mu\nu\alpha\beta}p_{0\nu}p_{+\alpha}p_{-\beta}\,F^{3\pi}(s_+,s_-,s_0)\,. \tag{3.41}$$

The current algebra result for this amplitude is denoted by $F_0^{3\pi}$ and reads

$$F_0^{3\pi} = \frac{-8e\kappa}{3F_\pi^3} \tag{3.42}$$

The tree-level contribution from the lagrangian $\mathcal{L}^W_{e^2p^4}$ to $F^{3\pi}$ is easily evaluated to be,

$$F^{3\pi}_{\rm tree} = \frac{-32e^3}{3F^3}\left(-2k^W_{2,r} + 2k^W_{3,r} + k^W_{4,r}\right). \tag{3.43}$$

Using the results above one finds that this part has the following scale dependence,

$$\mu\frac{d}{d\mu}F^{3\pi}_{\rm tree} = \frac{e^2}{16\pi^2}F_0^{3\pi}\left(\frac{-4C}{3F^4} + 3\right). \tag{3.44}$$

This scale dependence should cancel that arising from the one-loop contributions to the amplitude. According to ref. [10], however,

$$\mu\frac{d}{d\mu}F^{3\pi}_{\rm loop} = \frac{-e^2}{16\pi^2}F_0^{3\pi}\left(\frac{-8C}{F^4} + 3\right). \tag{3.45}$$

Indeed we find the cancellation occuring for the e^2 term, but not for the e^2C term.

4. Chiral and electromagnetic corrections to $\pi^0 \to 2\gamma$

We begin by setting up a complete list of the corrections to the current algebra result for the π^0 decay amplitude including both quark mass and electromagnetic corrections which one might, a priori, expect to be of comparable magnitude. An investigation with a similar scope was undertaken some time ago by Kitazawa [28]. Here, we will be using the approach of ChPT. In this framework, all corrections are expressed in terms of a minimal set of coupling-constants: at a given order, it is garanteed that no effect has been forgotten or double counted. We expect couplings from several sectors of ChPT to be involved: couplings from $\mathcal{L}_{p^4}$, from $\mathcal{L}^W_{p^6}$ (it is important here that the minimal number of independent couplings of this type has now been correctly determined [27, 29]) and, concerning EM corrections, couplings from $\mathcal{L}_{e^2p^2}$ and from $\mathcal{L}^W_{e^2p^4}$ that we have discussed above. In a next step, we will discuss resonance saturation estimates for all the combinations of couplings which are involved. Such estimates cannot be expected to be very accurate but they do provide reliable orders of magnitude and this discussion will show that, in fact, a single term dominates which one can then determine by making use of the ChPT expression for the η decay amplitude.

The T-matrix element for the π^0 decay into two photons has the following structure

$$\langle\gamma(p)\gamma(q)|\mathcal{T}|\pi^0\rangle = \epsilon^{\mu\nu\alpha\beta}p_\mu e_\nu q_\alpha e'_\beta\, A_{\pi2\gamma} \tag{4.1}$$

in terms of the momenta and the polarization vectors of the two photons. Let us call the current algebra result for the amplitude $A_{\rm CA}$. It has the well-known expression [7, 8]

$$A_{\rm CA} = \frac{\alpha}{\pi F_\pi}\,, \tag{4.2}$$

where F_π is the pion decay constant. Let us recall how this quantity is determined from experiment. As shown by Marciano and Sirlin [30] F_π can be related to the charged pion decays $\pi^+ \to \mu^+\nu, \mu^+\nu\gamma$ by the formula,

$$\begin{aligned}\Gamma(\pi^+ \to \mu^+\nu(\gamma)) = {} & F_\pi^2\,\frac{G_F^2|V_{ud}|^2}{4\pi} m_\mu^2 m_{\pi^+}\left(1-z_{\pi\mu}^2\right)^2 \left(1+\frac{2\alpha}{\pi}\log\frac{m_Z}{m_\rho}\right)\left(1+\frac{\alpha}{\pi}F(z_{\pi\mu})\right)\times \\ & \times\left(1-\frac{\alpha}{\pi}\left(-\frac{3}{4}\log\frac{m_\pi^2}{m_\rho^2}+c_1\right)\right)\end{aligned} \tag{4.3}$$

with $z_{\pi\mu} = m_\mu/m_{\pi^+}$ and the function F has the following expression

$$\begin{aligned}F(x) = {} & 3\log x + \frac{13-19x^2}{8(1-x^2)} - \frac{8-5x^2}{2(1-x^2)^2}\,x^2\log x - \\ & -2\left(\frac{1+x^2}{1-x^2}\log x + 1\right)\log(1-x^2) + 2\frac{1+x^2}{1-x^2}\int_0^{1-x^2}\frac{\log(1-t)}{t}dt\,.\end{aligned} \tag{4.4}$$

Further radiative corrections proportional to the ratio m_μ^2/m_ρ^2 have been dropped. In eq. (4.3) all the dependence upon the electric charge e^2 is displayed explicitly such that F_π is defined at $e^2 = 0$. In this situation, the difference between F_{π^+} and F_{π^0} is quadratic in $m_u - m_d$, it can be shown to be of the order of 10^{-4} and can be ignored for our purposes. The only undetermined parameter in eq. (4.3) is c_1. One can think of this parameter as collecting a number of ChPT low-energy coupling-constants.[2] In ref. [30] its variation with the scale was used to estimate that it must lie in a range $c_1 \simeq 0 \pm 2.4$. A more sophisticated analysis using resonance models for the various form factors involved was undertaken by Finkemeier [32] who obtains $c_1 = -3.0 \pm 0.8$. Chiral perturbation theory shows that, at one loop, there are electromagnetic effects which are induced by the $\pi^+ - \pi^0$ mass difference [31]. The corresponding chiral logarithms are not accounted for by the resonance models[3] and we feel that they could be added explicitly into c_1. In this way, together with that appearing in eq. (4.3) the complete set of chiral logarithms is included. Finally, we will use

$$c_1 = -3.0 \pm 0.8 + \frac{C}{4F_\pi^4}\left(3 + 2\log\frac{m_\pi^2}{m_\rho^2} + \log\frac{m_K^2}{m_\rho^2}\right), \tag{4.5}$$

where C is the coupling which appears at chiral order e^2 [5]. For the other quantities we use the values from the PDG [33], i.e.

$$\begin{aligned}\Gamma(\pi^+ \to \mu^+\nu(\gamma)) &= 0.25281(5)\,10^{-16}\ {\rm GeV}\,, \qquad V_{ud} = 0.9735(8) \\ G_F &= 1.16637(1)\,10^{-5}\ {\rm GeV}^{-2}\,.\end{aligned} \tag{4.6}$$

[2] An explicit expression in terms of such coupling-constants was derived in ref. [31].

[3] These models provide a leading large-N_c approximation as, for instance, the resonances are taken as infinitely narrow.

This results in the following value for F_π

$$F_\pi = 92.16 \pm 0.11\,\mathrm{MeV} \tag{4.7}$$

which we will use in what follows.[4]

Let us now consider the π^0 decay process from the point of view of ChPT. For this particular process, it will turn out to be fruitful to use the enlarged framework of the SU(3) chiral expansion. This expansion, of course, is less general than the SU(2) one, as it relies on the additional assumption that the strange quark mass is sufficiently small. In the SU(2) ChPT at $\mathrm{O}(p^4)$ the amplitude reads

$$A_{p^4} = \frac{\alpha}{\pi F}\,, \qquad [\mathrm{SU}(2)]\,, \tag{4.8}$$

where F is the pion decay constant in the limit $m_u = m_d = 0$. In the SU(3) expansion at $\mathrm{O}(p^4)$, the amplitude receives a correction due to the $\pi^0 - \eta$ mixing and reads

$$A_{p^4} = \frac{\alpha}{\pi F_0}\left(1 + \frac{m_d - m_u}{4(m_s - m)}\right), \qquad m = \frac{1}{2}(m_u + m_d)\,, \qquad [\mathrm{SU}(3)]\,, \tag{4.9}$$

where F_0 is the pion decay constant in the limit $m_u = m_d = m_s = 0$. At this level of the chiral expansion, one can identify F and F_0 with F_π but, still, the two amplitudes differ. For the problem at hand, the SU(3) expansion enables one to make use of input from the η decay amplitude and this will result in much improved predictions. Therefore, we will use the SU(3) chiral expansion in the following. Let us now list the contributions at the next chiral order.

1) One-loop meson diagrams which are one-particle irreducible.[5] An exact cancellation occurs between the tadpole and the unitarity contributions [34, 35]. This still holds when $\mathrm{O}(e^2)$ contributions are included in the masses.

2) One-particle irreducible tree contributions from $\mathcal{L}^W_{p^6}$. In SU(3), there are only two terms from the list of ref. [27] which contribute, C^W_7 and C^W_8,

$$\mathcal{L}^W_{p^6} = \epsilon^{\mu\nu\alpha\beta}\left\{\cdots + C^W_7\,\langle\chi_- f_{+\mu\nu} f_{+\alpha\beta}\rangle + C^W_8\,\langle\chi_-\rangle\,\langle f_{+\mu\nu} f_{+\alpha\beta}\rangle + \cdots\right\}. \tag{4.10}$$

It is convenient to introduce two related dimensionless parameters

$$T_1 = -\frac{256\pi^2}{3} m_\pi^2 C^W_7\,, \qquad T_1' = -\frac{1024\pi^2}{3} m_\pi^2 C^W_8 \tag{4.11}$$

(we use the same notation as ref. [36] the difference in the sign reflects the different convention for the epsilon tensor). If one were to perform the SU(2) chiral expansion, there would be four independent coupling constants involved: c^W_3, c^W_7, c^W_8 and c^W_{11}.

[4] The difference with the value quoted in the PDG [33], $F_\pi = 92.4 \pm 0.3$ MeV, comes partly from our using the value of c_1 from ref. [32] and partly from using the values of G_F and V_{ud} provided by the same PDG.

[5] The separate contributions of one-particle reducible and irreducible diagrams depends on the representations of the matrix U and the fluctuation matrix in terms of pion fields. For this matter, we will follow the conventions of section 3

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3) One-loop and tree contributions which are one-particle reducible. These can be expressed in terms of wave-function renormalization and mixing. Including electromagnetic contributions but neglecting terms which are quadratic in isospin breaking (i.e. $(m_u - m_d)^2$, e^4, $e^2(m_u - m_d)$) wave function renormalization has the following form

$$\begin{pmatrix} \pi^3 \\ \pi^8 \end{pmatrix} = \begin{pmatrix} 1 & -\epsilon_1 - e^2\delta_1 \\ \epsilon_2 + e^2\delta_2 & 1 \end{pmatrix} \begin{pmatrix} \frac{F_0}{F_\pi}(1 + e^2\delta_\pi)\pi^0 \\ \frac{F_0}{F_\eta}(1 + e^2\delta_\eta)\eta \end{pmatrix} . \tag{4.12}$$

The expressions for ϵ_1, ϵ_2 were given in ref. [4], we reproduce the formula for ϵ_2 here which is relevant for our calculations

$$\epsilon_2 = \frac{\sqrt{3}(m_d - m_u)}{4(m_s - m)} \Bigg\{ 1 - \frac{32(m_K^2 - m_\pi^2)}{F_0^2}(3L_7 + L_8) - 3\mu_\pi + 2\mu_K + \mu_\eta + \\ + \frac{m_\pi^2}{16\pi^2 F_0^2} \left(1 - \frac{m_\pi^2}{m_K^2 - m_\pi^2} \log \frac{m_K^2}{m_\pi^2} \right) \Bigg\} , \tag{4.13}$$

where

$$\mu_P = \frac{m_P^2}{32\pi^2 F_0^2} \log \frac{m_P^2}{\mu^2} , \qquad P = \pi,\ K,\ \eta . \tag{4.14}$$

The electromagnetic contributions which will be relevant for us are contained in δ_2 and δ_π in eq. (4.12) which have the following expressions

$$\begin{aligned} \delta_\pi &= -\frac{1}{2}C_1 + \frac{C}{F_0^4}(4\tilde{\nu}_\pi + \tilde{\nu}_K) , \\ \delta_2 &= \frac{\sqrt{3}\, m}{m_s - m} \left(\frac{3}{2}C_2 - \frac{2}{3}(K_9 + K_{10}) - \frac{2C}{F_0^4}\tilde{\nu}_K \right) , \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} C_1 &= \frac{8}{3}(K_1 + K_2) - 2(2K_3 - K_4) + \frac{20}{9}(K_5 + K_6) , \\ C_2 &= -\frac{2}{3}(2K_3 - K_4) + \frac{4}{9}(K_5 + K_6) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \tilde{\nu}_P &= \frac{1}{32\pi^2} \left(\log \frac{m_P^2 F_0^2 + 2e^2 C}{\mu^2 F_0^2} + \frac{m_P^2 F_0^2}{2e^2 C} \log \left(1 + \frac{2e^2 C}{m_P^2 F_0^2} \right) \right) \\ &\simeq \frac{1}{32\pi^2} \left(1 + \log \frac{m_P^2}{\mu^2} \right) . \end{aligned} \tag{4.17}$$

The last equality holds if one uses physical values for the quark masses and the electric charge. For completeness we also quote the expressions for the remaining two parameters δ_η and δ_1 in the mixing matrix

$$\begin{aligned} \delta_\eta &= -\frac{1}{2}C_1 + C_2 + \frac{3C}{F_0^4}\tilde{\nu}_K , \\ \delta_1 &= \frac{\sqrt{3}}{m_s - m} \left(\frac{1}{2}(2m_s + m)C_2 - \frac{2}{3}m(K_9 + K_{10}) - \frac{2C}{F_0^4} m_s \tilde{\nu}_K \right) . \end{aligned} \tag{4.18}$$

The electromagnetic contributions to $\pi - \eta$ mixing were previously discussed in ref. [37]. Their influence in the $\eta \to 3\pi$ decay rate was investigated in ref. [38].

A remark is in order here concerning the QCD renormalization group invariance of the combination $\epsilon_2 + e^2\delta_2$. The analysis of ref. [39] shows that $K_9 + K_{10}$ which appears in eq. (4.15) for δ_2 depends not only on the chiral scale μ but also on the QCD renormalization scale μ_0. However, ϵ_2 also is not RNG invariant because m_u and m_d have different charges. The scale dependence is proportional to $e^2 m/(m_s - m)$ and it can be seen to cancel out exactly in the combination $\epsilon_2 + e^2\delta_2$.

4) There are no one-loop diagrams with one photon in the loop. The last possible contributions are tree level contributions from $\mathcal{L}^W_{e^2p^4}$. Going through the list of independent terms eq. (2.29) one sees that none of them contributes to the π^0 decay amplitude. These terms correspond to the SU(2) expansion but this result is easily seen to hold for the SU(3) expansion as well.

Putting all this together, we can write the $\pi^0 \to 2\gamma$ amplitude including all the contributions which are of chiral order p^6 or e^2p^4, i.e. linear in the quark masses m_u, m_d and the electric charge e^2

$$A_{\pi 2\gamma} = A_{\rm CA}\left\{1 + \frac{\epsilon_2}{\sqrt{3}} + \frac{e^2\delta_2}{\sqrt{3}} + e^2\delta_\pi + \left(1 - \frac{m_d - m_u}{m}\frac{4r-5}{4(r-1)}\right)T_1 - \right.$$
$$\left. - \frac{3(m_d - m_u)}{4m}T_1'\right\}, \tag{4.19}$$

where r is the quark mass ratio

$$r = \frac{2m_s}{m_u + m_d}\,. \tag{4.20}$$

Altogether, there are five independent contributions which generate the deviation from the current algebra result. These contributions are of same chiral order but, in practice, they can have rather different sizes. This may be seen by using resonance models to estimate the orders of magnitude of the various chiral coupling constants which are involved. How this can be performed in practice is described in detail in appendix A. Using these results, one obtains the following estimates for the various entries in eq. (4.19)

$$\begin{aligned}
&\frac{\epsilon_2}{\sqrt{3}} \simeq 0.6\,10^{-2}\,, \qquad \frac{e^2\delta_2}{\sqrt{3}} \simeq -1.2\,10^{-6}\,, \qquad e^2\delta_\pi \simeq -0.3\,10^{-2}\,,\\
&|T_1| < 0.16\,10^{-2}\,, \qquad |T_1'| \simeq 3.04\,10^{-2}\,.
\end{aligned} \tag{4.21}$$

The term proportional to δ_2 is negligible and the one containing T_1 is small. The latter term is the only one which would survive in the isospin symmetric limit. The corrections are entirely dominated by isospin violation induced either by $m_u - m_d$ or by e^2. The dominant contribution in eq. (4.21) comes from T_1'. The resonance models do not determine the sign of this contribution. A more accurate determination including the sign can be performed based on the SU(3) ChPT expression at order p^6 of the $\eta \to 2\gamma$ decay width,

$$A_{\eta 2\gamma} = \frac{1}{\sqrt{3}}A_{\rm CA}\left\{\frac{F_\pi}{F_\eta} + \frac{5-2r}{3}T_1 + (1-r)T_1'\right\}. \tag{4.22}$$

The decay constant F_η which appears here is not known from experiment but can be determined using F_π, F_K which are known, together with ChPT [4]. Next, using the experimental result for the η decay width $\Gamma_{\eta 2\gamma} = 0.46 \pm 0.04\,\mathrm{KeV}$ and the fact that T_1 is much smaller than T_1' one can determine this parameter in terms of the quark mass ratio r (assuming that the sign of the amplitude is the same as its current algebra value),

$$T_1' = \frac{1}{1-r}(0.93 \pm 0.07 \pm 0.14)\,. \tag{4.23}$$

Here, the first error reflects the experimental error on the width and the second error reflects the uncertainty coming from higher order chiral corrections in the formula (4.22). We have assumed that such corrections do not to exceed 15%. One indication that higher order corrections are not large is that the value of T_1' obtained using eq. (4.22), $T_1' = (-3.7 \pm 0.3 \pm 0.4)\,10^{-2}$ (with the quark mass ratio taken to be $r = 26$) agrees with the estimate obtained using resonance models (see (4.21)). Using this evaluation of T_1' one can write the pion decay amplitude beyond current algebra as follows

$$A_{\pi 2\gamma} = A_{\mathrm{CA}}\left\{1 + \frac{m_d - m_u}{m_s - m}(0.93 \pm 0.12) - 0.34\,10^{-2} \pm 0.14\,10^{-2}\right\}. \tag{4.24}$$

In this formula, we have collected first the terms proportional to $m_d - m_u$ which involve the inverse of the quark mass ratio called R,

$$R = \frac{m_s - m_u}{m_d - m_u}\,. \tag{4.25}$$

The term which comes next is the electromagnetic contribution and the error (last term) collects the uncertainties in the estimate of this contribution and the uncertainty from T_1. The quark mass ratio R has been determined previously from several sources: a) leading order ChPT fit to the pseudo-scalar octet meson masses [40] (in particular the $K^0 - K^+$ mass difference) giving $R \simeq 43$, b) from $\rho\!-\!\omega$ mixing (e.g. [41] for a recent re-evaluation) and c) from the $\eta \to 3\pi$ decay in ChPT (see the discussion by Leutwyler [42]). The latter method seems to yield a slightly smaller value of R. Agreement of various determinations would imply that the expansion in the strange quark mass is rapidly converging and would eliminate the possibility that m_u is equal to zero (which gives a value of R of the order of twenty). In this sense it would be interesting if a precise measurement of the π^0 decay width could yield such a new determination upon using the formula (4.24). Unfortunately, the smallness of the effect and the theoretical uncertainty only allow for the order of magnitude of R to be tested. Using the value $R = 43$ the final theoretical prediction for the $\pi^0 \to 2\gamma$ decay width is

$$\Gamma_{\pi 2\gamma} = 8.06 \pm 0.02 \pm 0.06\,\mathrm{eV}\,. \tag{4.26}$$

The first error is associated with the determination of F_π (eq. (4.7)) and the second error collects the uncertainties in the evaluation of the chiral corrections. The overall accuracy is of the order of 1%.

5. Conclusions

In this paper, we have discussed the framework for including radiative corrections in chiral perturbation theory in the sector of the anomaly. For this purpose, we have first classified a minimal set of chiral lagrangian terms at $\mathrm{O}(e^2p^4)$. In the case of $N_f = 2$ there are eight such independent terms. Next, we have computed the divergence structure at this order. Contributions not only from irreducible but also from reducible one-loop graphs are present, which conforms to the general discussion of the $O(p^6)$ type divergences in ref. [23]. We have used a diagrammatic method which allows one to make use, eventually, of gauges more general than the Feynman gauge. The effective action at $O(e^2p^4)$ is non anomalous, similar to the results obtained previously for the other $O(p^6)$ components in the anomaly sector [24, 25, 26]. We expect that this property will no longer hold at higher order in e^2.

As a practical application, we have verified the divergence structure which was obtained from an explicit calculation of EM corrections to the $\gamma\pi \to \pi\pi$ amplitude: we find only partial agreement. As an other application, we have considered in some detail the π^0 decay amplitude which is of interest in view of forthcoming high-precision measurements at JLAB. In this case, the coupling-constants from $\mathcal{L}^W_{e^2p^4}$ turn out to make no contributions. Under the assumption that the SU(3) chiral expansion may be used, the number of coupling-constants involved remains tractable. In this case, the whole set of corrections to the current algebra result consists of five independent terms, accounting for both quark mass and electromagnetic corrections: this is the content of eq. (4.19). Estimates based on resonance models of the various chiral coupling-constants involved are reviewed and updated and a quantitative result is presented. The corrections are dominated by isospin breaking, with the electromagnetic contributions being small but not negligible.

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A. Resonance models of chiral coupling constants

The sizes of chiral coupling constants in ChPT can be understood, at a semi-quantitative level, in terms of properties of the light resonances. This was investigated in great detail by Ecker et al. [20] for the case of the couplings L_i of the $O(p^4)$ chiral lagrangian. In this approach one introduces first a lagrangian which includes, in addition to chiral fields and sources, resonances with proper transformation properties under the chiral group. The dynamics can be treated at tree level and one can match the Green's functions computed in this way with those obtained from ChPT. Let us discuss the predictions of such an approach for the set of chiral coupling constants relevant for the π^0 decay amplitude.

A.1 Couplings C_1, C_2

Let us consider first the Urech's couplings K_i. Quite generally, each of these couplings can be expressed as a QCD N-point Green's function (with N=2,3,4) convoluted with the photon propagator [39]. In some cases, informations on the relevant Green's functions can

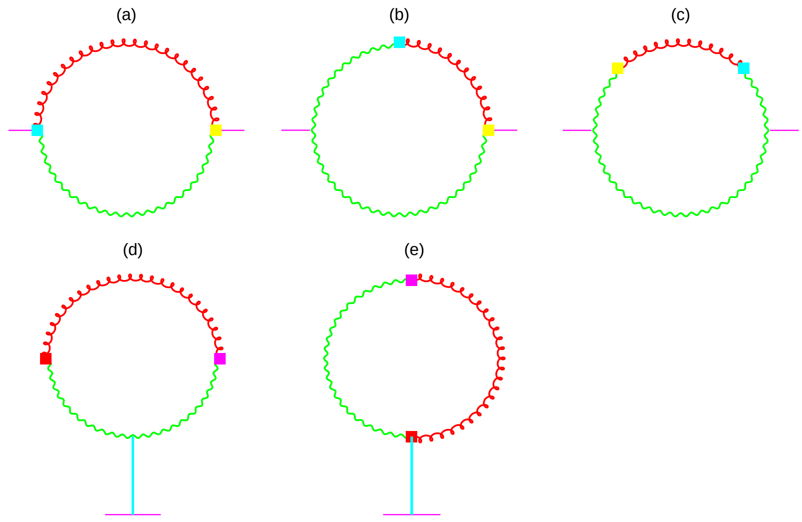

Figure 3: Graphs which contribute to order e^2 wave-function¡ renormalization in resonance models. Graphs (a), (b), (c) are generated by the vector meson lagrangian eq. (A.3). A curly line denotes a photon propagator and a wiggly line a vector meson propagator. Graphs (d), (e) show possible contributions from scalar resonances which we have not included.

be extracted from experiment. Otherwise, one may use models for these. Several papers have discussed estimates for some or combinations of $K_i's$ [39, 43, 44, 45] but we could not find a result for the specific combinations C_1, C_2 (see eq. (4.16)) which are needed here. In this case, a straightforward approach to obtain the resonance model estimate is the following.

One can easily identify the combinations C_1, C_2 by considering the effective action for the neutral pseudo-scalar mesons π^3, π^8 taken as slowly varying fields. In ChPT at order e^2p^2 the leading terms in an expansion in powers of the derivatives has the following form

$$\mathcal{S}_{\rm eff} = \frac{1}{2}e^2 \int d^4x\, \{K_{11}\nabla_\mu\pi^3\nabla^\mu\pi^3 + 2K_{12}\nabla_\mu\pi^3\nabla^\mu\pi^8 + K_{22}\nabla_\mu\pi^8\nabla^\mu\pi^8 + \cdots\} \tag{A.1}$$

with $\nabla_\mu\pi^i = \partial_\mu\pi^i - a^i_\mu$ and

$$\begin{aligned} K_{11} &= C_1 - \frac{2C}{F_0^4}(4\tilde\nu_\pi + \tilde\nu_K)\,, \\ K_{12} &= \sqrt{3}\left(C_2 - \frac{2C}{F_0^4}\tilde\nu_K\right), \\ K_{22} &= C_1 - 2C_2 - \frac{6C}{F_0^4}\tilde\nu_K\,. \end{aligned} \tag{A.2}$$

We will next generate the effective action for the neutral pseudo-scalars starting from the following lagrangian containing vector meson resonances [46],

$$\begin{aligned} \mathcal{L}_V = &-\frac{1}{4}\langle V_{\mu\nu}V^{\mu\nu} - 2m_V^2 V_\mu V^\mu\rangle - \frac{1}{2\sqrt{2}}f_V\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + \\ &+ \frac{g_1}{2\sqrt{2}}\,\epsilon^{\mu\nu\alpha\beta}\langle\{u_\mu, V_\nu\}f_{+\alpha\beta}\rangle + \frac{g_2}{2}\,\epsilon^{\mu\nu\alpha\beta}\langle\{u_\mu, V_\nu\}V_{\alpha\beta}\rangle\,. \end{aligned} \tag{A.3}$$

with $V_{\mu\nu} = \nabla_\mu V_\nu - \nabla_\nu V_\mu$. We use a description (introduced in ref. [46]) in terms of vector fields which transform homogeneously under the the non-linear representation of the chiral group,

$$V_\mu \longrightarrow h V_\mu h^\dagger \,. \tag{A.4}$$

This representation is related by a simple field transformation [46] to the hidden gauge approach [47]. As shown in ref. [46] different representations will give the same answer provided matching to the QCD asymptotic behaviour is imposed. There are three types of one-loop diagrams which have one photon in the loop, they are shown in figure 2. Computing these diagrams gives the following result for the effective action

$$S_{\rm eff} = \frac{4e^2}{F^2}\int d^4x \left\{ \frac{10}{9}\nabla_\mu\pi^3\nabla_\nu\pi^3 + \frac{4}{3\sqrt{3}}\nabla_\mu\pi^3\nabla_\nu\pi^8 + \frac{2}{3}\nabla_\mu\pi^8\nabla_\nu\pi^8 \right\} I^{\mu\nu} + \cdots \tag{A.5}$$

with

$$I^{\mu\nu} = i\int \frac{d^np}{(2\pi)^n}\frac{p^2 g^{\mu\nu} - p^\mu p^\nu}{p^2(p^2-m_V^2)^3}\left(p^2\left(g_1 + 2g_2 f_V\right) - g_1 m_V^2\right)^2 . \tag{A.6}$$

In the present case, matching to the QCD asymptotic behaviour is equivalent to requiring that the integral $I^{\mu\nu}$ converges (see [39]). This is satisfied provided the following relation holds

$$g_1 + 2g_2 f_V = 0 \,. \tag{A.7}$$

One can verify that the same relation ensures that the matrix element of the vector current between a vector meson and a pion satisfies vector meson dominance, i.e. has a simple pole form. Using eq. (A.7) one obtains

$$I^{\mu\nu} = g^{\mu\nu}\,\frac{3g_1^2 m_V^4}{128\pi^2} \,. \tag{A.8}$$

We must next match the effective action obtained in ChPT eq. (A.1) and the one obtained from the resonance model eq. (A.5). Here, we face a problem: the chiral effective action has chiral logarithms which are not present in the resonance model. This is due, in particular, to the use of infinitely narrow resonances in the model. A plausible resolution is to assume that the resonance model represents the part of the chiral action without the chiral logarithms, i.e. the part which contains the combination of couplings C_1, C_2 with the value of the scale set at $\mu = m_V$. One obtains, then

$$C_1 = \frac{5\,\tilde g_1^2}{24\pi^2}\,, \qquad C_2 = \frac{\tilde g_1^2}{24\pi^2}\,, \qquad \tilde g_1 = \frac{g_1 m_V}{F_\pi} \,. \tag{A.9}$$

The dimensionless coupling constant $\tilde g_1$ can be determined from the $\omega \to \pi\gamma$ decay width $\Gamma = 0.72 \pm 0.05\,\mathrm{MeV}$,

$$\Gamma = \alpha\,\frac{\tilde g_1^2 m_V}{6}\left(1 - \frac{m_\pi^2}{m_V^2}\right)^3 , \qquad \tilde g_1 = 0.91 \pm 0.03 \,. \tag{A.10}$$

In ref. [39] it was shown that the Urech's couplings $K_1, \ldots, K_6$ can be expressed as convolutions involving QCD four-point functions. The estimate discussed above corresponds

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to a vector meson pole model for the relevant four-point functions. Other possible poles, for instance involving scalar mesons (see figure 3), are ignored. In eq. (4.15) beside C_1 and C_2 one also needs the combination $K_9 + K_{10}$. We will use for this the resonance estimate given in ref. [39].

A.2 Couplings C_7^W, C_8^W

Let us now consider the two coupling-constants C_7^W, C_8^W from $\mathcal{L}_{p^6}^W$. For the sake of using a unified approach we review the estimates obtained using the same kind of resonance models as discussed above. The relevant resonances now are the pseudo-scalar ones, we consider an octet of these ($\pi(1300)$ family) and two singlets (corresponding to $\eta'(980)$ and $\eta(1295)$). Using the same notation as ref. [20] the couplings to the pseudo-scalar sources are contained in

$$\mathcal{L}_P = i d_m \langle P\chi_- \rangle + i\tilde{d}_m \eta_1 \langle \chi - \rangle + i\tilde{d}'_m \eta'_1 \langle \chi - \rangle \,. \tag{A.11}$$

In addition, we have to consider the couplings to two photons

$$\mathcal{L}_{P2\gamma} = \epsilon^{\mu\nu\alpha\beta}\{g_{\pi'} \langle P f_{+\mu\nu} f_{+\alpha\beta}\rangle + g_{\eta_1}\eta_1 \langle f_{+\mu\nu} f_{+\alpha\beta}\rangle + g_{\eta'_1}\eta'_1 \langle f_{+\mu\nu} f_{+\alpha\beta}\rangle\} \,. \tag{A.12}$$

Next, one has to integrate out the resonances to the order needed to generate terms belonging to $\mathcal{L}_{p^6}^W$. This is easily performed by making the field redefinitions

$$P \longrightarrow P + \frac{i d_m}{M_P^2}\left(\chi_- - \frac{1}{N_f}\langle \chi_- \rangle\right), \qquad \eta_1 \longrightarrow \eta_1 + \frac{i\tilde{d}_m}{M_{\eta_1}^2}\langle \chi_- \rangle \tag{A.13}$$

and similarly for η'_1. In this way, the following estimates of the couplings $C_{7,8}^W$ in terms of resonance parameters emerge,

$$C_7^W = \frac{g_{\pi'} d_m}{M_P^2}\,, \qquad C_8^W = \frac{g_{\eta_1}\tilde{d}_m}{M_{\eta_1}^2} + \frac{g_{\eta'_1}\tilde{d}'_m}{M_{\eta'_1}^2} - \frac{g_{\pi'} d_m}{N_f M_P^2}\,. \tag{A.14}$$

It is possible to express $C_{7,8}^W$ in a more general way. In the case of C_7^W, for instance, consider the matrix element of the pseudo-scalar current between two photon states,

$$\langle \gamma(p)|j_P^3(0)|\gamma(q)\rangle = \epsilon^{\mu\nu\alpha\beta} p_\mu e_\nu q_\alpha e'_\beta\, F_P(t)\,, \qquad t = (p-q)^2 \tag{A.15}$$

with $j_P^3 = i\bar{\psi}\gamma^5 \frac{\lambda^3}{2}\psi$. Then, it is possible to express C_7^W as an unsubtracted dispersive representation [28],

$$B_0 C_7^W = -\frac{3}{64\pi}\int_{9m_\pi^2}^\infty \frac{dt'\,\mathrm{Im}\,F_P(t')}{t'}, \tag{A.16}$$

an analogous representation also holds for C_8^W. These representations are convergent because asymptotically, the perturbative QCD contribution reads,

$$\mathrm{Im}\,F_P^{\mathrm{QCD}}(t) = -\frac{2}{3}\frac{\alpha m}{t}\log\frac{\sqrt{t}+\sqrt{t-4m^2}}{2m}(1+\mathrm{O}(\alpha_s)) \tag{A.17}$$

and convergence is even faster in the chiral limit $m = 0$. Therefore, the approximation to retain the lightest resonance contribution in the integrand which, in the narrow width approximation reproduces the results of the resonance model discussed above, should be quite reasonable. In the expression (A.14) for C_8^W the last two terms cancel exactly in the leading large-N_c limit and we expect the first contribution, from the η' to be largely dominant. The parameters $\tilde{d}_m$ and M_{η_1} have been estimated in ref. [20]

$$|\tilde{d}_m| \simeq 20\,\mathrm{MeV}\,, \qquad M_{\eta_1} \simeq 804\,\mathrm{MeV}\,. \tag{A.18}$$

There remains to evaluate the parameter, d_m, which was not done in ref. [20]. For this purpose, we may impose on the resonance model to match the QCD asymptotic behaviour for the Green's function

$$\Pi_{SP}(p^2) = i\int d^4x\ \exp(ipx)\,\langle 0|T(j_S^3(x)j_S^3(0) - j_P^3(x)j_P^3(0))|0\rangle \tag{A.19}$$

which is easily shown to go like $1/(p^2)^2$ in the chiral limit. This imposes the following Weinberg-type relation between d_m and its scalar counterpart c_m

$$d_m^2 + \frac{F_0^2}{8} = c_m^2\,. \tag{A.20}$$

In addition to that, we may use the relation between d_m, c_m and the low-energy coupling-constant L_8 (see [20]) in the same resonance saturation model,

$$L_8 = \frac{c_m^2}{2M_S^2} - \frac{d_m^2}{2M_P^2}\,. \tag{A.21}$$

Taking $L_8 \simeq 0.9\,10^{-3}$ gives,

$$|c_m| \simeq 51\,\mathrm{MeV}\,, \qquad |d_m| \simeq 39\,\mathrm{MeV}\,. \tag{A.22}$$

Next, the parameters $g_{\pi'}$, g_{η_1}, can be deduced from the two-photons decay rates of the $\pi(1300)$ and the η',

$$\Gamma_{\pi' 2\gamma} = \alpha^2 g_{\pi'}^2 \frac{128\pi}{9} m_{\pi'}^2\,, \qquad \Gamma_{\eta' 2\gamma} = \alpha^2 g_{\eta_1}^2 \frac{1024\pi}{9} m_{\eta'}^2\,. \tag{A.23}$$

The experimental values for these decay widths are [33, 48]

$$\Gamma_{\pi' 2\gamma} < 0.1\,\mathrm{KeV}\,, \qquad \Gamma_{\eta' 2\gamma} = 4.29 \pm 0.15\,\mathrm{KeV}\,. \tag{A.24}$$

This determines all the necessary ingredients in the expressions (A.14) for $C_{7,8}^W$ and one deduces the following result for the dimensionless quantities T_1 and T_1' proportional to C_7^W and C_8^W,

$$|T_1| < 0.16\,10^{-2}\,, \qquad |T_1'| \simeq 3.04\,10^{-2}\,. \tag{A.25}$$

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